CENTRALIZERS OF DISTINGUISHED NILPOTENT PAIRS AND RELATED PROBLEMS

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ABSTRACT. In this paper, by establishing an explicit and combinatorial description of the centralizer of a distinguished nilpotent pair in a classical simple Lie algebra, we solve in the classical case Panyushev's Conjecture which says that distinguished nilpotent pairs are wonderful, and the classification problem on almost principal nilpotent pairs. More precisely, we show that disinguished nilpotent pairs are wonderful in types A, B and C, but they are not always wonderful in type D. Also, as the corollary of the classification of almost principal nilpotent pairs, we have that almost principal nilpotent pairs do not exist in the simply-laced case and that the centralizer of an almost principal nilpotent pair in a classical simple Lie algebra is always abelian.

INTRODUCTION

The study of nilpotent pairs in semisimple Lie algebras is due to V. Ginzburg [2]. Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field of characteristic zero and let G be its adjoint group. Then a pair of commuting nilpotent elements $\mathbf{e} = (e_1, e_2)$ in $\mathfrak{g}$ is called a nilpotent pair if there exists a pair of commuting semisimple elements $\mathbf{h} = (h_1, h_2)$ having rational eigenvalues in the adjoint action such that $[h_i, e_j] = \delta_{i,j} e_j$. The pair $\mathbf{h}$ is called an associated semisimple pair (or a characteristic of $\mathbf{e}$ by D. Panyushev [3]). The theory of nilpotent pairs can be viewed as a double counterpart of the theory of nilpotent orbits and Ginzburg showed in [2] that the theory of principal nilpotent pairs, *i.e.* the simultaneous centralizer of $\mathbf{e}$ has dimension $\operatorname{rk}\mathfrak{g}$, has a refinement of Kostant's results on regular nilpotent elements in $\mathfrak{g}$.

Although the number of G-orbits of nilpotent pairs is infinite, it was shown in [2] that, as in the classical case, the number of G-orbits of principal nilpotent pairs is finite. Generalizing the notion of distinguished nilpotent element, Ginzburg also introduced the notion of distinguished nilpotent pairs which are defined to be those such that the simultaneous centralizer does not contain any semisimple elements and the simultaneous centralizer of an associated semisimple pair is a Cartan subalgebra, and he conjectured that the number of G-orbits of distinguished nilpotent pairs is finite. In [1], A.

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Elashvili and D. Panyushev gave a positive answer to this conjecture in the case of classical simple Lie algebras by giving an explicit classification of distinguished nilpotent pairs in these Lie algebras.

On the other hand, Panyushev [3] introduced the notion of wonderful nilpotent pairs (see Section 5 for the definition) and he gave a proof of the finiteness of the number of G-orbits of wonderful nilpotent pairs. In the pursuit of a classification-free approach to the finiteness of the number of G-orbits of distinguished nilpotent pairs, he conjectured that distinguished nilpotent pairs are wonderful.

In the first part of this paper, we give a complete description of the centralizer of a distinguished nilpotent pair for a classical simple Lie algebra $\mathfrak{g}$. This is done by generalizing the approach of Ginzburg [2] in the case of $\mathrm{sl}(V)$, using skew diagrams. This description is very explicit and our starting point is the classification of Elashvili and Panyushev.

The combinatorial tool introduced here allows us to study in a purely diagram-theoretic manner the dimension of the centralizer of a distinguished nilpotent pair. In particular, we prove that the dimension of its positive quadrant is greater than or equal to the rank of $\mathfrak{g}$, and we determine exactly when equality holds. An application of our results shows that a distinguished nilpotent pair is wonderful in type A, B or C, but it is only so in type D under certain (necessary and sufficient) conditions. Thus we answer the above conjecture completely in the classical case.

In [4], Panyushev studied another class of nilpotent pairs called almost principal. A nilpotent pairs is called almost principal if its centralizer has dimension $\operatorname{rk}\mathfrak{g} + 1$. He proved that an almost principal nilpotent pair is distinguished and wonderful and he gave examples of such nilpotent pairs in the non simply-laced cases. He suspected that almost nilpotent pairs do not exist in the simply-laced cases and suggested that the centralizer of an almost principal nilpotent pair is abelian.

Using our results on the centralizers of distinguished nilpotent pairs, we give a classification of almost principal nilpotent pairs. As corollaries, we confirm that there are indeed no almost principal nilpotent pairs in the simply-laced cases and also that the centralizer of an almost principal nilpotent is effectively abelian.

In Section 1, we define our combinatorial tool and recall the classification of Elashvili and Panyushev. In Section 2, we prove that the centralizer of a distinguished nilpotent pair has a basis indexed by a certain set of pairs of skew diagrams. We study in Section 3 and 4 the combinatorics of this set of pairs of skew diagrams. In Section 5, we give necessary and sufficient conditions for a distinguished nilpotent pair to be wonderful. Sections 6 and 7 are devoted to the classification of almost principal nilpotent pairs and to their centralizer.

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1. Generalities on distinguished nilpotent pairs

Let $\mathbf{e} = (e_1, e_2)$ be a nilpotent pair, and $\mathbf{h} = (h_1, h_2)$ an associated semisimple pair. The nilpotent pair $\mathbf{e}$ is called **distinguished** if its centralizer $Z(\mathbf{e}) := Z_{\mathfrak{g}}(e_1) \cap Z_{\mathfrak{g}}(e_2)$ in $\mathfrak{g}$ contains no semisimple element and the centralizer $Z(\mathbf{h}) := Z_{\mathfrak{g}}(h_1) \cap Z_{\mathfrak{g}}(h_2)$ is a Cartan subalgebra of $\mathfrak{g}$.

Before recalling the classification of distinguished nilpotent pairs in classical simple Lie algebras, we shall start by defining the combinatorial objects involved.

Definition 1.1. *By a* **diagram***, we mean a subset Γ of $\mathbb{R}^2$ such that there exist $x, y \in \mathbb{R}$ such that $\Gamma \subset \mathbb{Z}^2 + (x, y)$. It is called* **connected** *if given $(i, j), (k, l) \in \Gamma$, there exists a sequence of elements $(a_1, b_1), \cdots, (a_n, b_n)$ in Γ such that*

i) $(a_1, b_1) = (i, j)$ and $(a_n, b_n) = (k, l)$;
ii) for $1 \le t \le n-1$, $(a_{t+1} - a_t)^2 + (b_{t+1} - b_t)^2 = 1$.

A **skew diagram** *is a connected diagram Γ of $\mathbb{R}^2$ such that*

1. *Γ is finite;*
2. *if $(i, j) \in \Gamma$ and $(i+1, j+1) \in \Gamma$, then $(i+1, j)$ and $(i, j+1) \in \Gamma$.*

A subset Γ' of a skew diagram Γ is called a **skew subdiagram** *if Γ' is a skew diagram and if $(i, j) \in \Gamma'$, then $(i-1, j), (i, j-1) \notin \Gamma \setminus \Gamma'$.*

Remark 1.2. *Our definition corresponds to the usual definition of connected skew diagrams.*

Let $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ be the central symmetric bijection sending (i, j) to $(-i, -j)$.

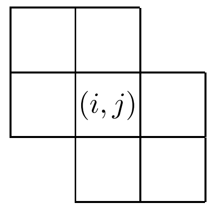
Definition 1.3. *A skew diagram Γ is* **centrally symmetric** *if $\Gamma = \sigma(\Gamma)$. A subset Γ' of a skew diagram Γ is called a* **σ-skew subdiagram** *if $\sigma(\Gamma')$ is a skew subdiagram of $\sigma(\Gamma)$.*

Note that there are three types of centrally symmetric skew diagrams:

Integral: $\Gamma \subset \mathbb{Z}^2$;
Semi-integral: $\Gamma \subset \mathbb{Z}^2 + (1/2, 0)$ *or* $(0, 1/2)$;
Non-Integral: $\Gamma \subset \mathbb{Z}^2 + (1/2, 1/2)$;

The notion of a σ-skew subdiagram will be used to describe the centralizer of $\mathbf{e}$. It corresponds to what Ginzburg called an *out-subset* in [2], while a skew subdiagram corresponds to an *in-subset*.

Example 1.4. *We shall represent a skew diagram by boxes. For example, the following is a skew diagram which is centrally symmetric if* $(i,j)=(0,0)$.



The subset $\{(i-1,j),(i,j),(i,j-1)\}$ *is a skew subdiagram and* $\{(i+1,j),$ $(i,j),$ $(i,j+1)\}$ *is a* σ*-skew subdiagram.*

Let us now recall the classification of distinguished nilpotent pairs in classical simple Lie algebras of A. Elashvili and D. Panyushev.

Theorem 1.5. [1] *The set of conjugacy classes of distinguished nilpotent pairs in* $\mathfrak{g}$ *is in bijection with:*

Type A - *the set of diagrams* Γ *of cardinal* $\operatorname{rk}\mathfrak{g}+1$ *such that*

- Γ *is a skew diagram;*
- *the barycentre of* Γ *is* $(0,0)$, *i.e.* $\sum_{(i,j)\in\Gamma} i=0$ *and* $\sum_{(i,j)\in\Gamma} j=0$

Type B - *the set of pairs of centrally symmetric skew diagrams* (Γ^1,Γ^2) *(we allow empty diagrams) such that*

- $\operatorname{Card}\Gamma^1+\operatorname{Card}\Gamma^2=2\operatorname{rk}\mathfrak{g}+1$;
- Γ^1 *is integral and* Γ^2 *is non integral.*

Type C - *the set of pairs of centrally symmetric skew diagrams* (Γ^1,Γ^2) *(we allow empty diagrams) such that*

- $\operatorname{Card}\Gamma^1+\operatorname{Card}\Gamma^2=2\operatorname{rk}\mathfrak{g}$;
- Γ^1 *and* Γ^2 *are semi-integral and* $\Gamma^1\subset\mathbb{Z}^2+(0,1/2)$, $\Gamma^2\subset\mathbb{Z}^2+(1/2,0)$.

Type D - *the set of triples* $(\Gamma^1,\Gamma^2,\Gamma^3)$ *of centrally symmetric skew diagrams and* $\epsilon\in\{1,2\}$ *(we allow empty diagrams) such that*

- $\operatorname{Card}\Gamma^1+\operatorname{Card}\Gamma^2+\operatorname{Card}\Gamma^3=2\operatorname{rk}\mathfrak{g}$;
- Γ^1 *is non integral;*
- *either* Γ^2 *and* Γ^3 *are empty or they are both non empty, integral with* $\Gamma^2\cap\Gamma^3=\{(0,0)\}\neq\Gamma^3$ *and* $\epsilon=1$.

We shall finish this section by expliciting the corresponding distinguished nilpotent pairs listed in Theorem 1.5. Let $(\Gamma^1,\Gamma^2,\Gamma^3)$ be as above, and

$$V=\bigoplus_{(i,j)\in\Gamma^1}\mathbb{C}v_{i,j}^1\oplus\bigoplus_{(i,j)\in\Gamma^2}\mathbb{C}v_{i,j}^2\oplus\bigoplus_{(i,j)\in\Gamma^3}\mathbb{C}v_{i,j}^3$$

Define $h_1,h_2\in\operatorname{gl}(V)$ by $h_1v_{i,j}^k=iv_{i,j}^k$ and $h_2v_{i,j}^k=jv_{i,j}^k$.
Type A, $\Gamma^2=\Gamma^3=\emptyset$, and define $e_1v_{i,j}^1=v_{i+1,j}^1$, $e_2v_{i,j}^1=v_{i,j+1}^1$. Then $\mathbf{e}=(e_1,e_2)$ is the corresponding distinguished nilpotent pair with $\mathbf{h}=(h_1,h_2)$ as the associated semi-simple pair.

Type B, $\Gamma^3 = \emptyset$, and define the non-degenerate symmetric bilinear form on V by

$$(v^n_{i,j}, v^m_{k,l}) = \delta_{i,-k}\delta_{j,-l}\delta_{n,m}.$$

Set

$$e_1 v^k_{i,j} = (-1)^{i+j} v^k_{i+1,j}, e_2 v^k_{i,j} = (-1)^{i+j+1} v^k_{i,j+1}.$$

Then $\mathbf{e} = (e_1, e_2)$ is the corresponding distinguished nilpotent pair with $\mathbf{h} = (h_1, h_2)$ as the associated semi-simple pair.

Type C, $\Gamma^3 = \emptyset$, and define the non-degenerate alternating bilinear form on V by

$$(v^n_{i,j}, v^m_{k,l}) = (-1)^{i+j+1/2}\delta_{i,-k}\delta_{j,-l}\delta_{n,m}.$$

Set

$$e_1 v^k_{i,j} = v^k_{i+1,j}, e_2 v^k_{i,j} = v^k_{i,j+1}.$$

Then $\mathbf{e} = (e_1, e_2)$ is the corresponding distinguished nilpotent pair with $\mathbf{h} = (h_1, h_2)$ as the associated semi-simple pair.

Type D, define the non-degenerate symmetric bilinear form on V by

$$(v^n_{i,j}, v^m_{k,l}) = \delta_{i,-k}\delta_{j,-l}\delta_{n,m}.$$

Set

$$e_1 v^k_{i,j} = (-1)^{i+j} v^k_{i+1,j}, e_2 v^k_{i,j} = (-1)^{i+j+1} v^k_{i,j+1}.$$

Then $\mathbf{e} = (e_1, e_2)$ is the corresponding distinguished nilpotent pair with $\mathbf{h} = (h_1, h_2)$ as the associated semi-simple pair.

Remark 1.6. *1. The construction for type C differs from the one in* [1]*. It seems that our construction is more adapted for our purpose since the definitions of the e_i's are much simpler.*

2. For type D, the parameter ϵ is not present here in the description. This is however not important concerning the description of the centralizer. The fact is that in this special case, there are two classes of distinguished nilpotent pairs admitting such a description (the corresponding bases for V are not conjugate).

2. Description of the centralizer of a distinguished nilpotent pair

Let $Z(\mathbf{e})$ be the centralizer of $\mathbf{e}$ in $\mathfrak{g}$. Note that it is h_i-stable for $i = 1, 2$. Thus it is $\mathbb{Q}^2$-graded and we shall denote by $Z_{p,q}(\mathbf{e})$ the homogeneous component of degree (p, q) which is nothing but the (h_1, h_2)-eigenspace of eigenvalue (p, q). Since $\mathbf{e}$ is distinguished, the centralizer $Z(\mathbf{h})$ of $\mathbf{h}$ in $\mathfrak{g}$ is a Cartan subalgebra. But $\mathfrak{g}_{0,0} = Z(\mathbf{h})$, so $Z_{0,0}(\mathbf{e}) = 0$ since $Z(\mathbf{e})$ contains no semi-simple elements.

Note that by Theorem 1.5, in fact $Z_{p,q}(\mathbf{e})$ is non zero only if p, q are half integers, *i.e.* $2p, 2q \in \mathbb{Z}$ and $p + q \in \mathbb{Z}$.

Let us conserve the notations in Section 1 and for $1 \leq k \leq 3$, set $V_k = \bigoplus_{(i,j)\in\Gamma^k} \mathbb{C}v^k_{i,j}$.

Let $x \in Z_{p,q}(\mathbf{e})$. When $k \neq l$, V_k and V_l are orthogonal with respect to the corresponding non-degenerate $\mathfrak{g}$-invariant bilinear form, it follows that if π_k denotes the canonical orthogonal projection, then x is the sum $\sum_{k,l} \pi_l \circ x \circ \pi_k$. For simplicity, let us fix k and l and assume that $x = \pi_l \circ x \circ \pi_k + \pi_k \circ x \circ \pi_l$ if $k \neq l$ and $x = \pi_k \circ x \circ \pi_k$ if $k = l$; *i.e.* $xV_k \subset V_l$, $xV_l \subset V_k$ and $xV_j = 0$ if $j \neq k, l$.

Definition 2.1. *We define* $\Gamma_x^k = \{(i,j) \in \Gamma^k$ *such that* $x.v_{i,j}^k \neq 0\}$.

We are interested in the connected components of Γ_x^k.

Lemma 2.2. *Let C be a connected component of Γ_x^k. Then,*

a) C is a skew subdiagram of Γ^k.
b) $C + (p,q)$ is a σ-skew subdiagram of Γ^l.

Proof. Let $(i,j) \in C$. If $(i-1,j)$ belongs to Γ^k, then $0 \neq e_1 v_{i-1,j}^k \in \mathbb{C} v_{i,j}^k$. Hence

$$e_1 x v_{i-1,j}^k = x e_1 v_{i-1,j}^k \neq 0$$

which implies that $(i-1,j) \in C$. The same argument applies to $(i,j-1)$ and part a) follows.

For b), again let $(i,j) \in C$. Thus $0 \neq x v_{i,j}^k \in \mathbb{C} v_{i+p,j+q}^l$. If $(i+p+1, j+q) \in \Gamma^l$, then $e_1 v_{i+p,j+q}^l \neq 0$. Hence

$$x e_1 v_{i,j} = e_1 x v_{i,j} \neq 0.$$

It follows that $x v_{i+1,j} \neq 0$, $(i+1,j) \in C$ and $(i+1+p, j+q) \in C + (p,q) \subset \Gamma^l$. The same argument applies to $(i+p, j+q+1)$ and we have proved that $C + (p,q)$ is a σ-skew subdiagram of Γ^l. □

Note that C and $C + (p,q)$ are of the same shape.

Denote by $\mathcal{C}_x^k$ the set of connected components of Γ_x^k.

Proposition 2.3. *Let $\mathfrak{g}$ be of type A, then $\Gamma = \Gamma^1$. For $C \in \mathcal{C}_x^1$, define x_C to be the obvious restriction of x to $V_C = \bigoplus_{(i,j)\in C} \mathbb{C} v_{i,j}^1$. Then $x_C \in Z_{p,q}(\mathbf{e})$ and $x = \sum_{C \in \mathcal{C}_x^1} x_C$.*

Proof. Since x commutes with the e_i's, it is clear that x_C is nilpotent and commutes also with the e_i's and so the first statement is clear. The last statement is also clear since we are taking connected components. □

For types B, C and D, the Γ^j's are centrally symmetric and we have the following lemma.

Lemma 2.4. *Let $\mathfrak{g}$ be of type other than A, then for any $C \in \mathcal{C}_x^k$, $\sigma(C) - (p,q) \in \mathcal{C}_x^l$.*

Proof. By Lemma 2.2, $C + (p,q)$ is a σ-skew subdiagram of Γ^l. Since Γ^l is centrally symmetric, $\sigma(C) - (p,q) = \sigma(C + (p,q))$ is a skew subdiagram of Γ^l.

Now let $(i,j) \in C$ and denote by $(\ ,\ )$, the $\mathfrak{g}$-invariant non-degenerate bilinear form corresponding to the type of $\mathfrak{g}$ as explained in Section 1. We have

$$(xv^k_{i,j}, v^l_{-i-p,-j-q}) + (v^k_{i,j}, xv^l_{-i-p,-j-q}) = 0.$$

Thus $xv^l_{-i-p,-j-q} \neq 0$ and $(-i-p,-j-q) \in \Gamma^l_x$.

To finish the proof, it suffices to show that $\sigma(C) - (p,q)$ is a connected component of Γ^l_x. Let $(i,j) \in \sigma(C) - (p,q)$ be such that $(i+1,j) \in \Gamma^l_x$. We have $0 \neq xv^l_{i+1,j} \in \mathbb{C}v^k_{i+1+p,j+q}$. By central symmetry, this means that $(-i-1-p,-j-q) \in \Gamma^k$. This in turn implies that $(-i-1-p,-j-q) \in C$ since $(-i-p,-j-q) \in C$ and C is a skew subdiagram. Thus $(i+1,j) \in \sigma(C) - (p,q)$.

The same argument applies for $(i,j+1)$ and this proves that $\sigma(C) - (p,q)$ is a connected component of Γ^l_x. □

Definition 2.5. *We define the set*

$$\mathcal{C}^{k,l}_x := \{\{C, C'\} \mid C \in \mathcal{C}^k_x, C' \in \mathcal{C}^l_x \text{ and } \sigma(C) = C' + (p,q)\}.$$

Remark 2.6. *First of all, note that we take subsets instead of pairs to avoid repetitions when $k = l$. Also it is clear that $\mathcal{C}^{k,l}_x$ and $\mathcal{C}^{l,k}_x$ are identical.*

Proposition 2.7. *Let $\mathfrak{g}$ be of type other than A, $\{C, C'\} \in \mathcal{C}^{k,l}_x$ and $x_{C,C'}$ the restriction of x to $\bigoplus_{(i,j)\in C} \mathbb{C}v^k_{i,j} \oplus \bigoplus_{(i,j)\in C'} \mathbb{C}v^l_{i,j}$. Then $x_{C,C'} \in Z_{p,q}(\mathbf{e})$ and $x = \sum_{\{C,C'\}\in \mathcal{C}^{k,l}_x} x_{C,C'}$.*

Proof. It is clear that $x_{C,C'}$ commutes with the e_i's and a direct computation shows that $x_{C,C'} \in \mathfrak{g}$. The second statement follows from Lemma 2.4 (which is also true if we exchange k et l) since we are considering connected components. □

Note that $x_{C,C'} = x_{C',C}$.

It is now clear how to define an indexing set for a set of generators of $Z_{p,q}(\mathbf{e})$.

Definition 2.8. *Let $\mathcal{E}_{p,q}(\mathbf{e})^k$ be the set of skew subdiagrams C in Γ^k such that $C' = C + (p,q)$ is a σ-skew subdiagram of Γ^k.*

For $k \neq l$, let $\mathcal{E}_{p,q}(\mathbf{e})^{k,l}$ be the set of pairs of skew subdiagrams (C_1, C_2) such that C_1 is a skew subdiagram of Γ^k, C_2 is a skew subdiagram of Γ^l and $C_1 + (p,q) = \sigma(C_2)$.

For $k = l$, p, q are integers. If $p+q$ is odd, then let $\mathcal{E}_{p,q}(\mathbf{e})^{k,k}$ be the set of subsets of skew subdiagrams $\{C_1, C_2\}$ in Γ^k such that $C_1 + (p,q) = \sigma(C_2)$. If $p+q$ even, then let $\mathcal{E}_{p,q}(\mathbf{e})^{k,k}$ be the set of subsets of skew subdiagrams $\{C_1, C_2\}$ in Γ^k such that $C_1 \neq C_2$ and $C_1 + (p,q) = \sigma(C_2)$.

Set $\mathcal{E}_{p,q}(\mathbf{e})$ to be $\mathcal{E}_{p,q}(\mathbf{e})^1$ if $\mathfrak{g}$ is of type A with $\Gamma = \Gamma^1$ and to be the disjoint union of the $\mathcal{E}_{p,q}(\mathbf{e})^{k,l}$'s with $k \leq l$ if $\mathfrak{g}$ is of type B, C or D.

Theorem 2.9. *For $(p,q) \neq (0,0)$, the subspace $Z_{p,q}(\mathbf{e})$ has a basis indexed by the set $\mathcal{E}_{p,q}(\mathbf{e})$.*

Proof. We shall first show that if $x \in \mathcal{E}_{p,q}(\mathbf{e})$ and $C_1 \in \mathcal{C}_x^1$ (resp. $\{C_1, C_2\} \in \mathcal{C}_x^{k,l}$), then the element x_{C_1} (resp. x_{C_1,C_2}) as defined in Propositions 2.3 (resp. Proposition 2.7) is uniquely determined by C_1 (resp. C_1, C_2) up to a scalar.

Let $e_1 v_{i,j}^k = a_{i,j}^k v_{i+1,j}^k$, $e_2 v_{i,j}^k = b_{i,j}^k v_{i+1,j}^k$ and if appropriate, we let $(v_{i,j}^k, v_{-i,-j}^k) = d_{i,j}^k$.

Let x be in $Z_{p,q}(\mathbf{e})$ with the appropriate property. For $(i,j) \in \Gamma_x^k$, $x v_{i,j}^k = \lambda_{i,j}^k v_{i+p,j+q}^l$.

Now let us fix k and l and let $(i,j) \in C_1 \in \mathcal{C}_x^{k,l}$. If $(i+1,j) \in C_1$, then $e_1 x_{C_1} v_{i,j}^k = x_{C_1} e_1 v_{i,j}^k$ implies that

$$a_{i+p,j+q}^l \lambda_{i,j}^k = \lambda_{i+1,j}^k a_{i,j}^k.$$

Similarly, if $(i, j+1) \in C_1$, then

$$b_{i+p,j+q}^l \lambda_{i,j}^k = \lambda_{i,j+1}^k b_{i,j}^k.$$

In the case of type A and C, $a_{i,j}^k = b_{i,j}^k = 1$ and we have

$$\lambda_{i+1,j}^k = \lambda_{i,j}^k = \lambda_{i,j+1}^k.$$

In the case of type B and D, $a_{i,j}^k = (-1)^{i+j} = -b_{i,j}^k$ and we have

$$\lambda_{i+1,j}^k = (-1)^{p+q} \lambda_{i,j}^k = \lambda_{i,j+1}^k.$$

Also, we have for types B, C and D,

$$(x v_{i,j}^k, v_{-i-p,-j-q}^l) + (v_{i,j}^k, x v_{-i-p,-j-q}^l) = 0$$

which implies that

$$\lambda_{i,j}^k d_{i+p,j+q}^l + \lambda_{-i-p,-j-q}^l d_{i,j}^k = 0.$$

When $k = l$, we have $p, q \in \mathbb{Z}$. If $p+q$ is even, then we must have $C_1 \neq C_2$ because from the classification of Section 1, $d_{i,j}^k = 1$ for types B, D and $d_{i,j}^k = (-1)^{i+j+1/2}$ for type C (one should treat the case where $C_1 = \{(i,j)\}$ separately).

Since p, q are fixed and C_1 is connected, we observe that the restriction of x on V_{C_1} and V_{C_2} are uniquely determined by the value of $\lambda_{i,j}^k$ (i, j, k are fixed here). It follows that x_{C_1} (resp. x_{C_1,C_2}) is uniquely determined by C_1 (resp. (C_1, C_2)) up to a scalar.

By Propositions 2.3 and 2.7, we have that $Z_{p,q}(\mathbf{e})$ is spanned by a set of elements indexed by a subset of $\mathcal{E}_{p,q}(\mathbf{e})$. To finish the proof of this theorem, il suffices to construct an element of $Z_{p,q}(\mathbf{e})$ for each element of $\mathcal{E}_{p,q}(\mathbf{e})$ and show that they form a basis of $Z_{p,q}(\mathbf{e})$. Recall that $Z_{p,q}(\mathbf{e})$ is non zero only if $2p, 2q \in \mathbb{Z}$ and $p+q \in \mathbb{Z}$.

Type A. Let Γ be a skew diagram with barycentre $(0,0)$ and $C \in \mathcal{E}_{p,q}(\Gamma)$. Define $x_C \in \mathrm{gl}(V)$ by

$$x_C v_{i,j} = \begin{cases} v_{i+p,j+q} & \text{if } (i,j) \in C \\ 0 & \text{otherwise.} \end{cases}$$

Then a simple verification shows that x_C commutes with e_1 and e_2. Further x_C is nilpotent since $(p,q) \neq (0,0)$ and therefore $x_C \in \mathrm{sl}(V)$.

Types B and D. Let Γ^k, Γ^l be centrally symmetric skew diagrams. Let $\{C, C'\} \in \mathcal{E}_{p,q}(\Gamma^k, \Gamma^l)$ be such that $k \neq l$ or $C \neq C'$. Define $x_{C,C'} \in \mathrm{gl}(V)$ by

$$x_{C,C'} v^m_{i,j} = \begin{cases} v^l_{i+p,j+q} & \text{if } (i,j) \in C, m = k \\ -v^k_{i+p,j+q} & \text{if } (i,j) \in C', m = l \\ 0 & \text{otherwise} \end{cases}$$

if $p+q$ is even; and if $p+q$ is odd, we set

$$x_{C,C'} v^m_{i,j} = \begin{cases} (-1)^{i+j} v^l_{i+p,j+q} & \text{if } (i,j) \in C, m = k \\ (-1)^{i+j} v^k_{i+p,j+q} & \text{if } (i,j) \in C', m = l \\ 0 & \text{otherwise.} \end{cases}$$

Again a simple verification shows that $x_{C,C'}$ commutes with e_1 and e_2, and that $x_{C,C'}$ is nilpotent since $(p,q) \neq (0,0)$. Finally, we verify easily that for $(i,j) \in C$,

$$(x_{C,C'} v^k_{i,j}, v^l_{-i-p,-j-q}) + (v^k_{i,j}, x_{C,C'} v^l_{-i-p,-j-q}) = 0.$$

Therefore $x_{C,C'} \in Z_{p,q}(\mathbf{e})$.

Now suppose that $l = k$ and $p+q$ is odd. Let $C \in \mathcal{E}_{p,q}(\Gamma^k)$ be such that $C = \sigma(C) - (p,q)$. Define $x_{C,C} \in \mathrm{gl}(V)$ by

$$x_{C,C} v^m_{i,j} = \begin{cases} (-1)^{i+j} v^k_{i+p,j+q} & \text{if } (i,j) \in C, m = k \\ 0 & \text{otherwise.} \end{cases}$$

The same verification works and therefore $x_{C,C} \in Z_{p,q}(\mathbf{e})$.

Type C. Let Γ^k, Γ^l be centrally symmetric skew diagrams. Let $\{C, C'\} \in \mathcal{E}_{p,q}(\Gamma^k, \Gamma^l)$ be such that $k \neq l$ or $C \neq C'$. Define $x_{C,C'} \in \mathrm{gl}(V)$ by

$$x_{C,C'} v^m_{i,j} = \begin{cases} v^l_{i+p,j+q} & \text{if } (i,j) \in C, m = k \\ (-1)^{p+q+1} v^k_{i+p,j+q} & \text{if } (i,j) \in C', m = l \\ 0 & \text{otherwise.} \end{cases}$$

Again a simple verification shows that $x_{C,C'}$ commutes with e_1 and e_2, and that $x_{C,C'}$ is nilpotent since $(p,q) \neq (0,0)$. Finally, we verify easily that for $(i,j) \in C$,

$$(x_{C,C'} v^k_{i,j}, v^l_{-i-p,-j-q}) + (v^k_{i,j}, x_{C,C'} v^l_{-i-p,-j-q}) = 0.$$

Therefore $x_{C,C'} \in Z_{p,q}(\mathbf{e})$.

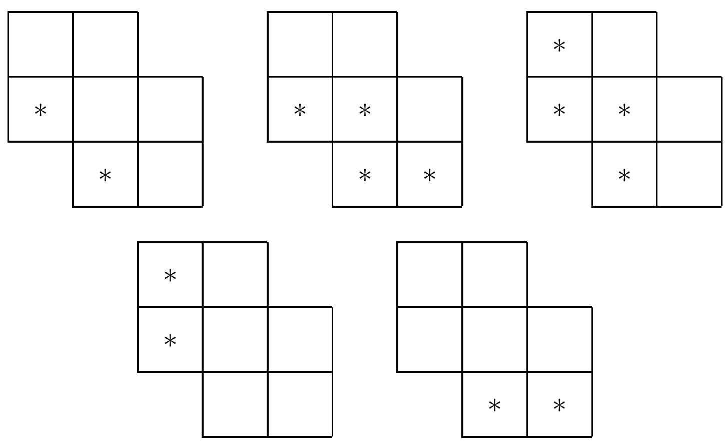
Now suppose that $l = k$ and $p+q$ is odd. Let $C \in \mathcal{E}_{p,q}(\Gamma^k)$ be such that $C = \sigma(C) - (p,q)$. Define $x_{C,C} \in \mathrm{gl}(V)$ by

$$x_{C,C} v^m_{i,j} = \begin{cases} v^l_{i+p,j+q} & \text{if } (i,j) \in C, m = k \\ 0 & \text{otherwise.} \end{cases}$$

The same verification works and therefore $x_{C,C} \in Z_{p,q}(\mathbf{e})$.

Thus we have constructed a set of elements of $Z_{p,q}(\mathbf{e})$ indexed by the set $\mathcal{E}_{p,q}(\mathbf{e})$. Finally, since they are defined on connected components or pairs of connected components, they are clearly linearly independant. The proof is now complete. □

Example 2.10. *Let us end this section by expliciting the centralizer of the distinguished nilpotent pair in* B_3 *of the centrally symmetric skew diagram* Γ *in Example 1.4. It has in fact a basis given by:*



where the "star" boxes denote the corresponding pairs of skew subdiagrams. Let x_1, x_2, x_3, x_4, x_5 *be the corresponding elements in* so_7 *(from left to right and top to bottom) defined by* $x_1 v_{0,-1} = v_{1,0}$, $x_1 v_{-1,0} = -v_{0,1}$ *and zero otherwise;* $x_2 v_{i,j} = (-1)^{i+j+1} v_{i,j+1}$, $x_3 v_{i,j} = (-1)^{i+j+1} v_{i+1,j}$, $x_4 v_{i,j} = (-1)^{i+j+1} v_{i+2,j-1}$, $x_5 v_{i,j} = (-1)^{i+j+1} v_{i-1,j+2}$ *for all* (i,j) *with* $v_{i,j} = 0$ *if* $(i,j) \notin \Gamma$. *A simple verification shows that this is a basis of* $Z(\mathbf{e})$.

3. Description of the positive quadrant via skew diagrams

Since the sets $\mathcal{E}_{p,q}(\mathbf{e})^k$ and $\mathcal{E}_{p,q}(\mathbf{e})^{k,l}$ depend only on the connected skew diagrams Γ^k and Γ^l, we can study it in a combinatorial way. For a skew diagram Γ, we define

- $\mathcal{E}_{p,q}(\Gamma)$ to be the set of skew subdiagrams C in Γ such that $C + (p,q)$ is a σ-skew subdiagram in Γ.

and for Γ, Γ' two distinct centrally symmetric skew diagrams, we define

- $\mathcal{E}_{p,q}(\Gamma, \Gamma')$ to be the set of pairs of skew subdiagrams (C, C') such that C is a skew subdiagram of Γ, C' is a skew subdiagram of Γ' and $C + (p, q) = \sigma(C')$.
- $\mathcal{E}_{p,q}(\Gamma, \Gamma)$ to be the set of subsets of skew subdiagrams $\{C, C'\}$ of Γ such that $C + (p, q) = \sigma(C')$ and if $p + q$ is even, then $C \neq C'$.

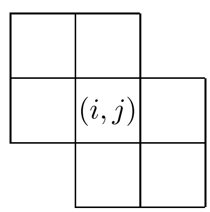
In this section, we are interested in the description of the positive quadrant, that is when $p, q \geq 0$. Let us fix $p, q \geq 0$. Note that if Γ, Γ' are both integral or non integral, then $p, q \in \mathbb{Z}$.

Lemma 3.1. *Let Γ be a skew diagram, then the cardinality of $\mathcal{E}_{p,q}(\Gamma)$ is the number of connected components of $(\Gamma - (p, q)) \cap \Gamma$. In particular $\mathcal{E}_{0,0}(\Gamma)$ is of cardinal* 1.

Proof. Since $p, q \geq 0$, it is clear that a connected component C of $(\Gamma-(p, q))\cap \Gamma$ is a skew subdiagram of Γ and that $C + (p, q)$ is a σ-skew subdiagram of Γ. $\square$

Let Γ be a skew diagram. An element $(i, j) \in \Gamma$ is called a northeast (resp. southwest) corner if $(i + 1, j)$ and $(i, j + 1)$ (resp. $(i - 1, j)$ and $(i, j - 1)$) are not in Γ. It is called a northeast (resp. southwest) angle if $(i + 1, j + 1)$ (resp. $(i - 1, j - 1)$) is not in Γ but $(i + 1, j)$ and $(i, j + 1)$ (resp. $(i - 1, j)$ and $(i, j - 1)$) are in Γ.

For example, let Γ be the following skew diagram,



then (i, j) is a southwest angle which is also a northeast angle, while $(i-1, j)$ and $(i, j - 1)$ are southwest corners, and $(i + 1, j)$, $(i, j + 1)$ are northeast corners.

Theorem 3.2. *Let Γ be a skew diagram of cardinal n. Then the cardinal of the set $\bigcup_{p,q\geq 0,(p,q)\neq(0,0)} \mathcal{E}_{p,q}(\Gamma)$ is equal to $n - 1$.*

Proof. First observe that $\mathcal{E}_{0,0}(\Gamma)$ has cardinal 1.

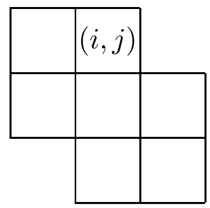
By the previous lemma, we are reduced to counting the number of connected components $e_{p,q}(\Gamma)$ of $(\Gamma - (p, q)) \cap \Gamma$. We shall proceed by induction on the cardinality of Γ. The result is trivial if the cardinality of Γ is 1.

Now let $n > 1$ be the cardinality of Γ. Let us first suppose that there exist a northeast corner $(i, j) \in \Gamma$ such that $\Gamma' = \Gamma \setminus \{(i, j)\}$ is again a skew diagram.

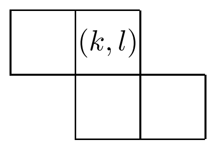
We shall compare the connected components of $(\Gamma' - (p, q)) \cap \Gamma'$ and $(\Gamma - (p, q)) \cap \Gamma$. Note that the former is obtained from the latter by removing $(i - p, j - q)$ set-theoretically. It follows by an easy observation that,

1. $e_{p,q}(\Gamma') = e_{p,q}(\Gamma) + 1$ if $(i,j) - (p,q)$ is a southwest angle of Γ', which is also a southwest angle of Γ.
2. $e_{p,q}(\Gamma') = e_{p,q}(\Gamma) - 1$ if $(i,j) - (p,q)$ is a southwest corner of Γ', which is also a southwest corner of Γ.
3. $e_{p,q}(\Gamma') = e_{p,q}(\Gamma)$ otherwise.

For example to illustrate 1, let Γ be the following skew diagram,



If we remove (i,j) and take $(p,q) = (0,1)$, then $(\Gamma - (p,q)) \cap \Gamma$ looks like



where $(k,l) = (i, j-1)$ is a southwest angle of Γ. So when we remove this box, we recover $(\Gamma' - (p,q)) \cap \Gamma'$ and the number of connected components is increased by 1.

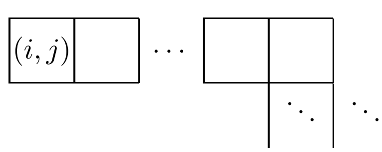
We are therefore reduced to counting southwest corners and southwest angles of Γ southwest of (i,j). Let N_c be the number of southwest corners (k,l) in Γ such that $k \leq i$ and $l \leq j$, and N_a the number of southwest angles (k,l) in Γ such that $k \leq i$ and $l \leq j$. Then it follows from the above that

$$\sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma) = \sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma') + N_c - N_a.$$

Now Γ is a skew diagram, so each southwest angle is above a southwest corner. Further, the leftmost southwest corner is not below any southwest angle. We conclude by induction that

$$\sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma) = \sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma') + 1 = n - 1.$$

We are left with the case where removing a northeast corner in Γ gives 2 skew diagrams. This means that Γ is of the form:



Let (i,j) be the leftmost, *i.e.* i minimal, southwest corner of Γ, and let $\Gamma' = \Gamma \setminus \{(i,j)\}$. Then it is clear that $e_{p,q}(\Gamma) = e_{p,q}(\Gamma')$ if $q > 0$. Let $(i+r, j)$

be the leftmost northeast corner of Γ. Then

$$e_{r,0}(\Gamma) = \begin{cases} e_{r,0}(\Gamma') & \text{if } p \neq r, \\ e_{r,0}(\Gamma') + 1 & \text{if } p = r. \end{cases}$$

It follows that $\sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma) = \sum_{p,q\geq 0,(p,q)\neq(0,0)} e_{p,q}(\Gamma') + 1 = n - 1$. □

Let us now turn to the study of $\mathcal{E}_{p,q}(\Gamma,\Gamma)$.

Lemma 3.3. *Let Γ be a centrally symmetric skew diagram, and let C be an element of $\mathcal{E}_{p,q}(\Gamma)$. Set $C' = \sigma(C) - (p,q)$. Then $C' = C$ or $C \cap C' = \emptyset$.*

Proof. First of all, we note that $C' + (p,q)$ is a σ-skew subdiagram of Γ, and since Γ is centrally symmetric, C' is a skew subdiagram of Γ. Thus $C' \in \mathcal{E}_{p,q}(\Gamma)$.

Let us suppose that $C \cap C'$ is non empty and C is not a subset of C'. Then there exists $(i,j) \in C \setminus C'$ such that $(i-1,j)$ or $(i,j-1)$ is in C'. This is not possible because $(i+p,j+q) \in \Gamma \setminus C' + (p,q)$, while $(i-1+p,j+q)$ or $(i+p,j-1+q)$ is in $C' + (p,q)$ which, as noted above, is a σ-skew subdiagram. The lemma now follows. □

Remark 3.4. *It follows from the previous lemma that $\mathcal{E}_{p,q}(\Gamma)$ is the disjoint union of those C such that $C = C'$ and subsets $\{C, C'\}$ with $C \cap C' = \emptyset$, where $C' = \sigma(C) - (p,q)$. Further, by central symmetry, there is at most one C in $\mathcal{E}_{p,q}(\Gamma)$ such that $C = C'$.*

Theorem 3.5. *Let Γ be a centrally symmetric skew diagram of cardinal n, then the cardinality of $\bigcup_{p,q\geq 0,(p,q)\neq(0,0)} \mathcal{E}_{p,q}(\Gamma,\Gamma) = \bigcup_{p,q\geq 0} \mathcal{E}_{p,q}(\Gamma,\Gamma)$ is equal to $[n/2]$, where $[n/2]$ denotes the integer part of $n/2$.*

Proof. First observe that $(C,C') \in \mathcal{E}_{0,0}(\Gamma,\Gamma)$ implies that $C = C'$. But $p+q$ is even, so $\mathcal{E}_{0,0}(\Gamma,\Gamma) = \emptyset$.

We proceed by induction on n as in Theorem 3.2. For $n = 1, 2$, the result is clear. Let $n > 2$ and let (i,j) be the northeast corner of Γ such that i is minimal. Let $e_{p,q}(\Gamma,\Gamma)$ denote the cardinality of $\mathcal{E}_{p,q}(\Gamma,\Gamma)$.

Let us first take care of the case where Γ is rectangular, *i.e.* Γ has exactly one southwest corner since Γ is centrally symmetric. Then every $C \in \mathcal{E}_{p,q}(\Gamma)$ is rectangular and $C = \sigma(C) - (p,q)$. So we have only to count the number of p,q's such that $p+q$ is odd. This is clearly $[n/2]$.

So let us suppose that Γ is not rectangular.

Let $\Gamma' = \Gamma \setminus \{(i,j), (-i,-j)\}$ and let us suppose that Γ' is again a skew diagram.

We shall use a similar argument as in Theorem 3.2, but since we are dealing with centrally symmetric skew diagrams, we need to have a detailed analysis on the induction process.

Denote by $\Delta_{p,q} = (\Gamma - (p,q)) \cap \Gamma$ and $\Delta'_{p,q} = (\Gamma' - (p,q)) \cap \Gamma'$. Note that $\Delta'_{p,q} = \Delta_{p,q} \setminus \{(-i,-j), (i-p, j-q)\}$.

Let C be a connected component of $\Delta_{p,q}$, then we have the following 3 lemmas:

Lemma 3.6. *We have $C \cap \{(-i,-j),(i-p,j-q)\} = \emptyset$ if and only if C is a connected component of $\Delta'_{p,q}$.*

Lemma 3.7. *Let $C \cap \{(-i,-j),(i-p,j-q)\} = \{(i-p,j-q)\}$ and $(-i,-j) \neq (i-p,j-q)$, then we have $C \neq \sigma(C)-(p,q)$, $(\sigma(C)-(p,q)) \cap \{(-i,-j),(i-p,j-q)\} = \{(-i,-j)\}$ and*

a) $C \setminus \{(i-p,j-q)\} = \emptyset$ if and only if $(i-p,j-q)$ is a southwest corner of Γ southwest of (i,j).

b) $C \setminus \{(i-p,j-q)\} = C'$ is a connected component of $\Delta'_{p,q}$ if and only if $(i-p,j-q)$ is neither a southwest corner nor a southwest angle of Γ southwest of (i,j). In particular, $\sigma(C')-(p,q) \neq C'$.

c) $C \setminus \{(i-p,j-q)\} = C' \cup C''$ is the disjoint union of two distinct connected components of $\Delta'_{p,q}$ if and only if $(i-p,j-q)$ is a southwest angle of Γ southwest of (i,j). In particular, $\sigma(C')-(p,q) \neq C'$, $\sigma(C'')-(p,q) \neq C''$ and $\sigma(C')-(p,q) \neq C''$.

Lemma 3.8. *Let $C \cap \{(-i,-j),(i-p,j-q)\} = \{(-i,-j),(i-p,j-q)\}$, then $C = \sigma(C)-(p,q)$ and*

a) $C \setminus \{(-i,-j),(i-p,j-q)\} = \emptyset$ if and only if either $(-i,-j) \in \{(i-p,j-q),(i-p-1,j-q)\}$ or $(-i,-j) = (i-p,j-q-1)$ and $(i-p,j-q)$ is not a southwest angle of Γ southwest of (i,j).

b) $C \setminus \{(-i,-j),(i-p,j-q)\} = C' \cup C''$ is the disjoint union of two distinct connected components of $\Delta'_{p,q}$ such that $C' = \sigma(C'')-(p,q)$ if and only if either $(-i,-j) = (i-p-1,j-q-1)$ or $(-i,-j) = (i-p,j-q-1)$ and $(i-p,j-q)$ is a southwest angle of Γ southwest of (i,j).

c) $C \setminus \{(-i,-j),(i-p,j-q)\} = C' \cup C'' \cup C'''$ is the disjoint union of three distinct connected components of $\Delta'_{p,q}$ such that $C' = \sigma(C'')-(p,q)$ and $C''' = \sigma(C''')-(p,q)$ if and only if $(i-p,j-q)$ is a southwest angle of Γ southwest of (i,j) such that $-i = i-p$ and $j-q-1 \neq -j$.

d) Otherwise $C \setminus \{(-i,-j),(i-p,j-q)\} = C'$ is a connected component of $\Delta'_{p,q}$ such that $C' = \sigma(C')-(p,q)$.

All three lemmas are easy consequences of Lemma 3.3, of the central symmetry of Γ and the fact that (i,j) is chosen to be the northeast corner with i minimal.

It follows from these three lemmas that $e_{p,q}(\Gamma,\Gamma)$ and $e_{p,q}(\Gamma',\Gamma')$ are equal unless (p,q) satisfies one of the following conditions:

1. $(i-p,j-q) \neq (-i,-j)$ is a southwest corner of Γ southwest of (i,j), in which case $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')+1$ by Lemma 3.7 a).
2. $(i-p,j-q)$ is a southwest angle of Γ southwest of (i,j) such that $i-p \neq -i$, in which case $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')-1$ by Lemma 3.7 c).

3. $(i-p, j-q)$ is a southwest angle of Γ southwest of (i,j) such that $i-p=-i$ and $j-q \neq -j+1$, in which case $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma') - 1$ by Lemma 3.8 b).
4. $(i-p, j-q) = (-i, -j)$ with $i \geq 0$, in which case
   i) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ if $2(i+j)$ is even by Lemma 3.8 a);
   ii) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')+1$ if $2(i+j)$ is odd by Lemma 3.8 a).
5. $(i-p, j-q) = (-i+1, -j+1)$ with $i > 0$, in which case
   i) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma') - 1$ if $2(i+j)$ is even by Lemma 3.8 b);
   ii) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ if $2(i+j)$ is odd by Lemma 3.8 b).
6. $(i-p, j-q) = (-i+1, -j)$ with $i > 0$, in which case
   i) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma') + 1$ if $2(i+j)$ is even by Lemma 3.8 a);
   ii) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ if $2(i+j)$ is odd by Lemma 3.8 a).
7. $(i-p, j-q) = (-i, -j+1)$ is a southwest angle of Γ southwest of (i,j), in which case
   i) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ if $2(i+j)$ is even by Lemma 3.8 b);
   ii) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma') - 1$ if $2(i+j)$ is odd by Lemma 3.8 b).
8. $(i-p, j-q) = (-i, -j+1)$ is not a southwest angle of Γ southwest of (i,j) and $i \geq 0$, in which case
   i) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma') + 1$ if $2(i+j)$ is even by Lemma 3.8 a);
   ii) $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ if $2(i+j)$ is odd by Lemma 3.8 a).

Now if $i < 0$, then only 1 and 2 apply, thus we deduce as in Theorem 3.2 that

$$\sum_{p,q\geq 0} e_{p,q}(\Gamma,\Gamma) = \sum_{p,q\geq 0} e_{p,q}(\Gamma',\Gamma') + N_c - N_a$$

where N_c (resp. N_a) is the number of southwest corners (resp. angles) southwest of (i,j).

Now if $i = 0$, then 5 and 6 do not apply. We have therefore

$$\sum_{p,q\geq 0} e_{p,q}(\Gamma,\Gamma) = \sum_{p>0,q\geq 0} e_{p,q}(\Gamma,\Gamma) + \sum_{q\geq 0} e_{0,q}(\Gamma,\Gamma).$$

There are 4 cases: $2(i+j)$ is even or odd; $(-i,-j+1)$ is or is not a southwest angle of Γ. By applying 3,4,7 and 8 in the appropriate case, we have that:

$$\sum_{q\geq 0} e_{0,q}(\Gamma,\Gamma) = \sum_{q\geq 0} e_{0,q}(\Gamma',\Gamma')$$

since Γ is not rectangular, so there is always a southwest angle above $(-i,-j)$. It follows that we have again

$$\sum_{p,q\geq 0} e_{p,q}(\Gamma,\Gamma) = \sum_{p,q\geq 0} e_{p,q}(\Gamma',\Gamma') + N_c - N_a.$$

Finally if $i > 0$, then 5 and 6 apply also. Again there are the same 4 cases to consider, and we deduce that

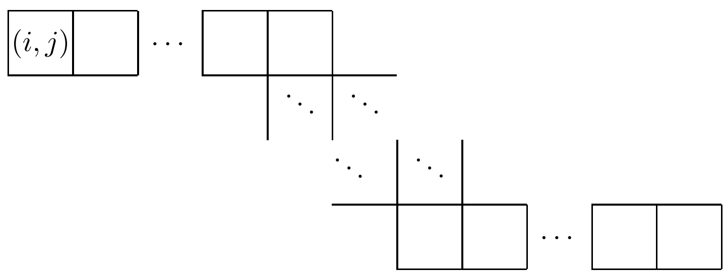
$$\sum_{0\leq p\leq 2i, q\geq 0} e_{p,q}(\Gamma,\Gamma) = \sum_{0\leq p\leq 2i, q\geq 0} e_{p,q}(\Gamma',\Gamma')$$

since for these (p,q), $e_{p,q}(\Gamma,\Gamma) = e_{p,q}(\Gamma',\Gamma')$ unless 3,4,5,6,7 or 8 is satisfied (note that $(-i,-j)$ is the first southwest corner of Γ southwest of (i,j) because $i > 0$ and (i,j) is chosen to be the leftmost northeast corner of Γ). Hence, we have again

$$\sum_{p,q\geq 0} e_{p,q}(\Gamma,\Gamma) = \sum_{p,q\geq 0} e_{p,q}(\Gamma',\Gamma') + N_c - N_a.$$

By induction, we have our result.

We are therefore left with the case where Γ' is not connected. In this case, let (i,j) be the leftmost southwest corner of Γ. Then Γ is of the following form:



A similar argument as in Theorem 3.2 reduces Γ to a horizontal diagram which is a simple verification. $\square$

Theorem 3.9. *Let Γ, Γ' be two centrally symmetric skew diagrams such that $\Gamma \cap \Gamma' = \{(0,0)\} \neq \Gamma'$. Then* $\mathrm{Card} \bigcup_{p,q\geq 0,(p,q)\neq(0,0)} \mathcal{E}_{p,q}(\Gamma,\Gamma') \geq 1$. *Equality holds if and only if Γ and Γ' satisfy one of the following two conditions:*

a) $\Gamma = \{(0,0)\}$ and Γ' has exactly one southwest corner (i,j) such that $i,j \leq 0$;

b) Γ and Γ' are both rectangular, i.e. they both have exactly one southwest corner and one northeast corner.

Proof. Let us first suppose that $\Gamma = \{(0,0)\}$. Then Γ' has at least a northeast corner (p,q) such that $p,q \geq 0$, $(p,q) \neq (0,0)$ since Γ' is a skew diagram of cardinal > 1 containing $(0,0)$. Thus the pair $\{(0,0)\}$, $\{(-p,-q)\}$ is in $\mathcal{E}_{p,q}(\Gamma,\Gamma')$.

We can therefore suppose that Γ and Γ' are skew diagrams of cardinal > 2 and $\Gamma \cap \Gamma' = \{(0,0)\}$. There exist $p,q > 0$ such that, exchanging the roles of Γ and Γ' if necessary, $(-p,0)$ be a southwest corner of Γ and $(0,q)$ be a northeast corner of Γ'. By central symmetry, $(0,-q) \in \Gamma'$ and the pair $\{(-p,0)\},\{(0,-q)\}$ is in $\mathcal{E}_{p,q}(\Gamma,\Gamma')$. So the first part of the theorem is proved.

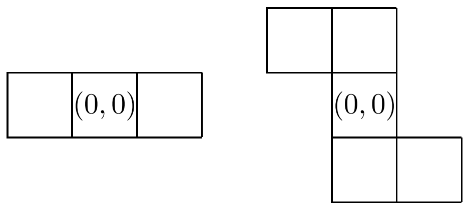
If $\Gamma = \{(0,0)\}$, then it is clear that equality holds if and only if condition a) is satisfied. So we can assume that the cardinality of Γ and Γ' are > 2.

If they are both rectangular, then one is horizontal and the other vertical. It is clear that equality holds. Now let Γ be non rectangular and that Γ contains $(-1,0)$ (the case Γ contains $(0,-1)$ is similar). Then Γ' contains $(0,-1)$.

Let $(0,q)$ be a northeast corner of Γ', then $q>0$ and $(0,q-1)\in\Gamma'$. Let $(-p,0)$ be a southwest corner of Γ with $p>0$. Then either $(-p-1,1)$ is not in Γ or Γ has a southwest corner $(-r,1)$ with $r>0$.

In the first case, the pair $\{(-p,0),(-p,1)\}$, $\{(0,-q),(0,-q+1)\}$ is in $\mathcal{E}_{p,q}(\Gamma,\Gamma')$. In the second case, the pair $(-r,1),(0,-q)$ is in $\mathcal{E}_{r,q-1}(\Gamma,\Gamma')$. Since the pair $(-p,0),(0,-q)$ is in $\mathcal{E}_{p,q}(\Gamma,\Gamma')$ in both cases, we are done. $\square$

Example 3.10. *Let* $\Gamma=\{(-1,0),(0,0),(1,0)\}$ *and* $\Gamma'=\{(-1,1),(0,1),(0,0),\ (0,-1),\ (1,-1)\}$ *be centrally symmetric skew diagrams.*



Then $\mathcal{E}_{1,1}(\Gamma,\Gamma')$ *contains the unique pair* $(-1,0),(0,-1)$ *and* $\mathcal{E}_{0,1}(\Gamma,\Gamma')$ *contains the unique pair* $\{(-1,0),(0,0)\},\{(0,-1),(1,-1)\}$. *One verifies easily that the cardinality of the set* $\bigcup_{p,q\geq 0}\mathcal{E}_{p,q}(\Gamma,\Gamma')$ *is* 2.

Proposition 3.11. *Let* Γ, Γ' *be centrally symmetric skew diagrams of different types. Then* $\bigcup_{p,q\geq 0,(p,q)\neq(0,0)}\mathcal{E}_{p,q}(\Gamma,\Gamma')$ *is non empty.*

Proof. A centrally symmetric skew diagram has always a southwest corner (i,j) such that $i,j\leq 0$. Let (i,j) (resp. (k,l)) be such a southwest corner in Γ (resp. Γ'). Then it is clear that the pair $(i,j),(k,l)$ is in $\mathcal{E}_{-i-k,-j-l}(\Gamma,\Gamma')$ and we are done. $\square$

4. Conditions (Y), (R) and Near rectangular skew diagrams

Let Γ be a skew diagram (resp. centrally symmetric skew diagram) and set

- $\mathcal{E}(\Gamma):=\bigcup_{(p,q)\in\mathbb{Z}^2}\mathcal{E}_{p,q}(\Gamma)$ (resp. $\mathcal{E}(\Gamma,\Gamma):=\bigcup_{(p,q)\in\mathbb{Z}^2}\mathcal{E}_{p,q}(\Gamma,\Gamma)$)
- and $\mathcal{E}_+(\Gamma):=\bigcup_{p,q\geq 0}\mathcal{E}_{p,q}(\Gamma)$ (resp. $\mathcal{E}_+(\Gamma,\Gamma):=\bigcup_{p,q\geq 0}\mathcal{E}_{p,q}(\Gamma,\Gamma)$).

From the previous section, we have that $\operatorname{Card}\mathcal{E}_+(\Gamma)=\operatorname{Card}\Gamma-1$ (resp. $\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)=[\operatorname{Card}\Gamma/2]$). In this section, we shall study skew diagrams (resp. centrally symmetric skew diagrams) such that the cardinality of $\mathcal{E}(\Gamma)$ (resp. $\mathcal{E}(\Gamma,\Gamma)$) is less than or equal to $1+\operatorname{Card}\mathcal{E}_+(\Gamma)$. (resp. $1+\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)$).

Definition 4.1. *We say that a skew diagram* Γ *satisfies (Y) if it has either exactly* 1 *southwest corner or exactly* 1 *northeast corner.*

So in the terminology of [2], Γ satisfies (Y) if and only if Γ is a Young diagram or a minus Young diagram, while in the terminology of [1], this is equivalent to that Γ is either sw-Young or ne-Young.

Proposition 4.2. *Let Γ be a skew diagram. Then*

a) $\mathcal{E}(\Gamma) = \mathcal{E}_+(\Gamma)$ *if and only if* Γ *satisfies (Y).*
b) if Γ *does not satisfy (Y), then* $\operatorname{Card}\mathcal{E}(\Gamma) \geq 2 + \operatorname{Card}\mathcal{E}_+(\Gamma)$.

Proof. First note that Γ has exactly one southwest corner is equivalent to $\sigma(\Gamma)$ has exactly one northeast corner, and that if $C \in \mathcal{E}_{p,q}(\Gamma)$, then $\sigma(C) - (p,q) \in \mathcal{E}_{p,q}(\sigma(\Gamma))$. So we can suppose that Γ has exactly one southwest corner (i,j). Now any skew subdiagram of Γ contains (i,j). So $\mathcal{E}_{p,q}(\Gamma)$ is empty if $p < 0$ or $q < 0$, and the "if" part follows.

Now suppose that Γ does not satisfy (Y), and therefore there are at least 2 southwest corners and at least 2 northeast corners. Let the bottom row of Γ be $(i,j), \cdots, (i+r,j)$ and the top row of Γ be $(k,l), \cdots, (k+s,l)$ where $r, s \geq 0$. Then Γ does not satisfy (Y) implies that $k < i$, $j < l$ and $i + r > k + s$.

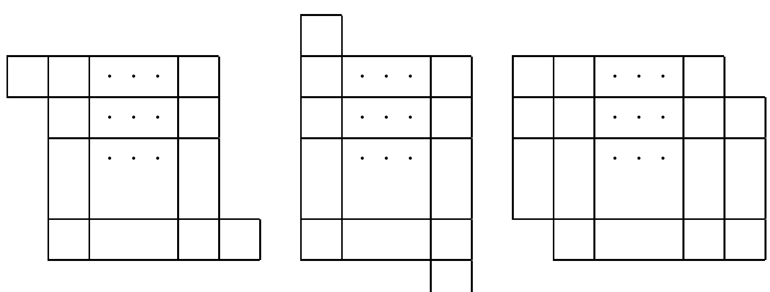
Now let $t = \min(r,s)$, then $k + s - t < i$. It follows that the skew diagram $\{(i,j), \cdots, (i+t,j)\}$ is in $\mathcal{E}_{k+s-t-i,l-j}(\Gamma)$. So part a) follows since $k + s - t - i < 0$.

Finally, if Γ does not satisfy (Y), then the above can also be applied to the leftmost column and the rightmost column. Thus b) follows. $\square$

Definition 4.3. *A centrally symmetric skew diagram Γ is called* **near rectangular** *if it has exactly 2 southwest corners (i,j), (k,l) with $i < k$ such that one of the following conditions is satisfied:*

a) $\Gamma \setminus \{(i,j), (-i,-j)\}$ *is rectangular;*
b) $\Gamma \setminus \{(k,l), (-k,-l)\}$ *is rectangular;*
c) $(i,j) = (k-1, l+1)$.

Their standard forms are illustrated as follows:



a) *b)* *c)*

We say that a skew diagram Γ satisfies (R) if either Γ is rectangular or Γ is non integral and near rectangular.

Proposition 4.4. *Let Γ be a centrally symmetric skew diagram. Then $\mathcal{E}(\Gamma,\Gamma)=\mathcal{E}_+(\Gamma,\Gamma)$ if and only if Γ satisfies (R).*

Proof. Let us suppose that Γ satisfies (R). If Γ is rectangular, then we have our result by Proposition 4.2. So let us suppose that Γ is near rectangular with southwest corners (i,j), (k,l) with $i<k$.

If Γ satisfies condition a) of Definition 4.3, then $k=i+1$ and $l=-j$. We observe that we have only to find a necessary and sufficient condition for $\{(i,j)\}$ to be in $\mathcal{E}_{-2i,-2j}(\Gamma,\Gamma)$ or $\{(i+1,-j),\cdots,(-i,-j)\}$ to be in $\mathcal{E}_{-1,2j}(\Gamma,\Gamma)$. This is so if and only if $2(i+j)$ or $2j-1$ is odd. But this is equivalent to saying that Γ is not non integral.

If Γ satisfies condition b) of Definition 4.3, then a similar argument gives our result.

Now if Γ satisfies condition c) of Definition 4.3, then $(k,l)=(i+1,j-1)$. We now have to find a necessary and sufficient condition for $\{(i,j),\cdots(i,-j+1)\}$ to be in $\mathcal{E}_{-2i,-1}(\Gamma,\Gamma)$ or $\{(i+1,j-1),\cdots,(-i,j-1)\}$ to be in $\mathcal{E}_{-1,-2j+2}(\Gamma,\Gamma)$. This is so if and only if $2i+1$ or $2j+1$ is odd. Again this is equivalent to saying that Γ is not non integral.

We have therefore proved the "if" part.

To prove the "only if" part, it suffices therefore to prove that if $\mathcal{E}(\Gamma,\Gamma)=\mathcal{E}_+(\Gamma,\Gamma)$, then Γ is either rectangular or near rectangular.

Let $(i,j),\cdots,(i+r,j)$ be the top row of Γ with $-i,j,r\geq 0$, then $(-i-r,-j),\cdots,(-i,-j)$ is the bottom row of Γ. Note that $i\leq -i-r$ since Γ is a skew diagram. If $r\geq 1$, then either $\{(-i-r,-j),\cdots,(-i,-j)\}$ is in $\mathcal{E}_{2i+r,2j}(\Gamma,\Gamma)$ or $\{(-i-r,j),\cdots,(-i-1,j)\}$ is in $\mathcal{E}_{2i+r+1,2j}(\Gamma,\Gamma)$ since either $2i+r+2j$ or $2i+r+2j+1$ is odd. So it follows that if $r\geq 1$, then $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $p<0$, $q>0$ if $2i+r+1<0$, or equivalently, $i+1<-i-r$.

Now let $(k,l),\cdots,(k,l+s)=(i,j)$ be the leftmost column of Γ with $-k,-l,s\geq 0$, then $(-k,-l-s),\cdots,(-k,-l)$ is the rightmost column of Γ. Note that $l\geq -l-s$ since Γ is a skew diagram. A similar argument shows that if $s\geq 1$, then $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $q<0$, $p>0$ if $-l-s<l-1$.

We are therefore left with four possibilities: $r=0$ and $s=0$; $r=0$ and $s>0$; $r>0$ and $s=0$; $r>0$ and $s>0$.

In the first case, we have that $\Gamma=\{(0,0)\}$ which is rectangular and the result follows.

In the second case, we have that $l-1\leq -l-s$. Since $l\geq -l-s$, we have either $l=-l-s$ or $l-1=-l-s$. If $l=-l-s$, then Γ is just a vertical diagram which is rectangular. If $l-1=-l-s$, then Γ is near rectangular satisfying condition b) of Definition 4.3.

The third case is similar to the second yielding condition a) of Definition 4.3. The last case implies that either $l=-l-s$, $i=-i-r$ or $l-1=-l-s$, $i+1=-i-r$. In the former, Γ is rectangular. In the latter, Γ is near rectangular satisfying condition c) of Definition 4.3.

Our proposition now follows. ∎

Proposition 4.5. *Let* Γ *be a centrally symmetric skew diagram. Then* $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma) = 1 + \operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)$ *if and only if one of the following conditions is satisfied:*

i) Γ *is semi-integral and near rectangular;*
ii) Γ *is integral and near rectangular of type a) or b).*

Proof. The "if" part is just a direct verification using the computations in the proof of Proposition 4.4. Let us show why type c) is not allowed for Γ integral and leave the other verifications to the reader. Let Γ be integral and near rectangular and (i,j), $(i+1,j-1)$ be the two distinct southwest corners with $i<k$. Then $2i+1$ and $2j+1$ are both odd, so by the computations in the proof of Proposition 4.4, there are 2 elements of $\mathcal{E}(\Gamma,\Gamma)$ which are not in $\mathcal{E}_+(\Gamma,\Gamma)$. Note that if Γ is near rectangular and non integral, then $\mathcal{E}(\Gamma,\Gamma)=\mathcal{E}_+(\Gamma,\Gamma)$.

To prove the "only if" part, it suffices to prove that if $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma) = 1+\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)$, then Γ is near rectangular. Let us suppose that Γ is not near rectangular.

Let $(i,j),\cdots,(i+r,j)$ be the top row of Γ with $-i,j,r\geq 0$, then $(-i-r,-j),\cdots,(-i,-j)$ is the bottom row of Γ. Note that $i\leq -i-r$ since Γ is a skew diagram.

Let $(k,l),\cdots,(k,l+s)=(i,j)$ be the leftmost column of Γ with $-k,-l$, $s\geq 0$, then $(-k,-l-s),\cdots,(-k,-l)$ is the rightmost column of Γ. Note that $l\geq -l-s$ since Γ is a skew diagram.

Recall from the proof of Proposition 4.4 that if $r\geq 1$ and $i+1<-i-r$, then $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $p<0$, $q>0$; and if $s\geq 1$ and $-l-s<l-1$, then $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $p>0$, $q<0$.

As in Proposition 4.4, we have four cases to consider: $r=s=0$; $r=0$, $s>0$; $s=0$, $r>0$; $r,s>0$.

In the first case, Γ is rectangular. So $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma)=\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)$.

In the second case, if $s=-2l$, then Γ is rectangular and $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma) = \operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)$. Since Γ is not near rectangular, $-l-s<l-1$ and $k<0$. So $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $p>0$ and $q<0$ if $-l-s<l-1$.

Let (u,v) be a southwest corner distinct from (k,l) and $(-i,-j)$. Then clearly, $i=k<u<-i$. It follows that $\{(u,v)\},\{(-i,-j)\}$ is a pair in $\mathcal{E}_{i-u,j-v}(\Gamma,\Gamma)$. Therefore $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma)\geq \operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)+2$.

So let us assume that (k,l) and $(-i,-j)$ are the only southwest corners of Γ. Then since $-l-s<l-1$, $(k+1,l+s-1)\notin\Gamma$ and by central symmetry $(-i-1,-j+1)\notin\Gamma$; otherwise, $(-i-1,-j+1)$ is a southwest corner distinct from (k,l) and $(-i,-j)$. It follows that either $\{(-i,-j)\}$ is in $\mathcal{E}_{k+i,l+s+j}(\Gamma,\Gamma)$ or $\{(-i,-j),(-i,-j+1)\}$ is in $\mathcal{E}_{k+i,l+s+j-1}(\Gamma,\Gamma)$, where $k+i<0$. Thus $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma)\geq\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)+2$.

The third case is similar to the second. Finally, in the last case, we have either $i+1\geq -i-r$, $-l-s<l-1$ or $i+1<-i-r$, $-l-s\geq l-1$. Without loss of generality, we can assume that $i+1\geq -i-r$, $-l-s<l-1$. Then

$i+r=-i$ or $i+r=-i-1$. Since $-l-s<l-1$, Γ is not rectangular and therefore we can not have $i+r=-i$. So $i+r=-i-1$ and Γ has exactly two southwest corners $(k,l)=(i,j-s)$ and $(k+1,-l-s)=(i+1,-j)$.

Now $\mathcal{E}_{p,q}(\Gamma,\Gamma)$ is non empty for some p,q with $p>0$ and $q<0$ since $-l-s<l-1$. Further, $-l-s<l-1$ implies that $(i,-j+1)\notin\Gamma$, therefore the rectangular diagram formed by the bottom two rows $\{(i+1,-j),\cdots,(-i,-j),(i+1,-j+1),\cdots,(-i,-j+1)\}$ is a skew subdiagram of Γ. It follows that either the bottom row $\{(i+1,-j),\cdots,(-i,-j)\}$ is in $\mathcal{E}_{k-i-1,l+s+j}(\Gamma,\Gamma)$ or $\{(i+1,-j),\cdots,(-i,-j),(i+1,-j+1),\cdots,(-i,-j+1)\}$ is in $\mathcal{E}_{k-i-1,l+s-1+j}(\Gamma,\Gamma)$ where $k-i-1=-1<0$. Again we conclude that $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma)\geq\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)+2$.

We have therefore proved that if Γ is not near rectangular, then we have $\operatorname{Card}\mathcal{E}(\Gamma,\Gamma)\neq\operatorname{Card}\mathcal{E}_+(\Gamma,\Gamma)+1$. Our proposition now follows. □

5. Panyushev's conjecture

We shall now apply the results of the previous sections to answer Panyushev's conjecture which states that distinguished nilpotent pairs are wonderful.

Let us denote by $Z_+(\mathbf{e})$ the direct sum $\bigoplus_{p,q\geq 0} Z_{p,q}(\mathbf{e})$.

Theorem 5.1. *The dimension of the positive quadrant $Z_+(\mathbf{e})$ is greater than or equal to the rank of $\mathfrak{g}$. Equality holds if and only if one of the following conditions is satisfied:*

a) $\mathfrak{g}$ is of type A;

b) $\mathfrak{g}$ is of type B and the associated pair of centrally symmetric skew diagrams (see Theorem 1.5) (Γ^1,Γ^2) satisfies $\Gamma^2=\emptyset$.

c) $\mathfrak{g}$ is of type C and the associated pair of centrally symmetric skew diagrams (see Theorem 1.5) (Γ^1,Γ^2) satisfies $\Gamma^1=\emptyset$ or $\Gamma^2=\emptyset$.

d) $\mathfrak{g}$ is of type D and the associated triple of centrally symmetric skew diagrams (see Theorem 1.5) $(\Gamma^1,\Gamma^2,\Gamma^3)$ is such that either $\Gamma^2=\Gamma^3=\emptyset$ or Γ^1 is empty and the pair (Γ^2,Γ^3) satisfies the conditions a) or b) of Theorem 3.9.

Proof. This is a direct consequence of the Theorem 2.9, Theorems 3.2, 3.5, 3.9 and Proposition 3.11. □

Definition 5.2. [3] *Recall that a nilpotent pair $\mathbf{e}=(e_1,e_2)$ is called* **wonderful** *if* $\dim\bigoplus_{p,q\in\mathbb{Z},p,q\geq 0} Z_{p,q}(\mathbf{e})=\operatorname{rk}\mathfrak{g}$.

It follows from Theorems 2.9, 3.2, 3.5 and 3.9 that:

Corollary 5.3. *A distinguished nilpotent pair $\mathbf{e}$ in a classical simple Lie algebra $\mathfrak{g}$ is wonderful if and only if one of the following conditions is satisfied:*

a) $\mathfrak{g}$ is of type A,B or C;

b) $\mathfrak{g}$ *is of type D and the associated triples of centrally symmetric skew diagrams (see Theorem 1.5)* $(\Gamma^1, \Gamma^2, \Gamma^3)$ *is such that the pair* (Γ^2, Γ^3) *satisfies the conditions a) or b) of Theorem 3.9.*

Proof. It suffices to observe that Proposition 3.11 does not apply here since we are only interested in $p, q \in \mathbb{Z}$. ☐

So distinguished nilpotent pairs are always wonderful in types A, B and C, but they are not so in type D. Therefore Panyushev's conjecture is not true in type D.

Example 5.4. *An example of a distinguished nilpotent pair which is not wonderful is the one described in Example 3.10 which corresponds to a distinguished nilpotent pair in* D_4*. One verifies easily that the positive quadrant has dimension* 5.

6. Classification of principal and almost principal nilpotent pairs

By applying the Theorem 5.1 and Propositions 4.2, 4.4, we recover easily the classification of principal nilpotent pairs.

Corollary 6.1. *Let* $\mathbf{e}$ *be a distinguished nilpotent pair in a classical simple Lie algebra* $\mathfrak{g}$*. Then* $\dim Z(\mathbf{e}) = \operatorname{rk} \mathfrak{g}$ *if and only if*

a) $\mathfrak{g}$ *is of type A and the associated skew diagram* Γ *satisfies (Y);*
b) $\mathfrak{g}$ *is of type B and the associated pair of centrally symmetric skew diagrams* $(\Gamma^1, \emptyset)$ *satisfies (R);*
c) $\mathfrak{g}$ *is of type C and the associated pair of centrally symmetric skew diagrams* $(\Gamma^1, \emptyset)$ *(resp.* $(\emptyset, \Gamma^2)$*) satisfies (R);*
d) $\mathfrak{g}$ *is of type D and the associated triples of centrally symmetric skew diagrams* $(\emptyset, \Gamma^2, \Gamma^3)$ *(resp.* $(\Gamma^1, \emptyset, \emptyset)$*) satisfies (R).*

The above corollary is exactly the classification of principal nilpotent pairs in classical simple Lie algebra given in [1].

Remark 6.2. *Note that there exist non principal distinguished nilpotent pairs satisfying* $Z(\mathbf{e}) = Z_+(\mathbf{e})$*. By the results of Sections 3 and 4, such a nilpotent pair must be of type other than A and the corresponding skew diagrams must all satisfy (R).*

For example, in type B and C, take any pair (Γ^1, Γ^2) *such that they are both non empty and rectangular. Then one can verify easily that* $Z(\mathbf{e}) = Z_+(\mathbf{e})$.

Also in type B, take any pair (Γ^1, Γ^2) *such that* Γ^1 *is rectangular with southwest corner* (i, j) *and* Γ^2 *is near rectangular satisfying for all* $(k, l) \in \Gamma^2$*, we have* $k > i$*,* $l > j$*, then again one can verify easily that* $Z(\mathbf{e}) = Z_+(\mathbf{e})$.

We now turn to the classification of almost principal nilpotent pairs.

Definition 6.3. *Recall that a nilpotent pair* $\mathbf{e}$ *is called* **almost principal** *if* $\dim Z(\mathbf{e}) = \operatorname{rk} \mathfrak{g} + 1$.

It is shown in [4] that almost principal nilpotent pairs are distinguished and wonderful. Therefore the number of G-orbits of almost principal nilpotent pairs is finite.

Theorem 6.4. *Let $\mathfrak{g}$ be of type A or D, then there are no almost principal nilpotent pairs.*

Proof. For type A, this is a direct consequence of Theorem 2.9 and Proposition 4.2.

So let us suppose that $\mathfrak{g}$ is of type D, and let $(\Gamma^1, \Gamma^2, \Gamma^3)$ be the corresponding triple of centrally symmetric skew diagram. If they are all non empty, then by Proposition 3.11, $\mathcal{E}_+(\Gamma^1, \Gamma^2)$ and $\mathcal{E}_+(\Gamma^1, \Gamma^3)$ are not empty where $\mathcal{E}_+(\Gamma^1, \Gamma^i) = \bigcup_{p,q\geq 0} \mathcal{E}_{p,q}(\Gamma^1, \Gamma^i)$. So $\operatorname{Card} Z(\mathbf{e}) \geq \operatorname{rk} \mathfrak{g} + 2$.

If $\Gamma^2 = \Gamma^3 = \emptyset$, then since Γ^1 is non integral, $\operatorname{Card} Z(\mathbf{e}) \neq \operatorname{rk} \mathfrak{g} + 1$ by Proposition 4.5.

Finally, suppose that $\Gamma^1 = \emptyset$. Then the condition $\operatorname{Card} Z(\mathbf{e}) = \operatorname{rk} \mathfrak{g} + 1$ would imply that by

$$\operatorname{Card} \mathcal{E}(\Gamma^2, \Gamma^2) + \operatorname{Card} \mathcal{E}(\Gamma^3, \Gamma^3) + \operatorname{Card} \mathcal{E}(\Gamma^2, \Gamma^3) = \operatorname{rk} \mathfrak{g} + 1$$

where $\mathcal{E}(\Gamma^2, \Gamma^3) = \bigcup_{(p,q)\in\mathbb{Z}^2} \mathcal{E}_{p,q}(\Gamma^2, \Gamma^3)$. So by Corollary 6.1, we deduce that either Γ^2 or Γ^3 does not satisfy (R). Without loss of generality, we can suppose that Γ^2 does not satisfy (R). Thus we have by Theorem 3.5 and Proposition 4.4 that

$$\operatorname{Card} \mathcal{E}(\Gamma^2, \Gamma^2) = 1 + \operatorname{Card} \mathcal{E}_+(\Gamma^2, \Gamma^2), \operatorname{Card} \mathcal{E}(\Gamma^3, \Gamma^3) = \operatorname{Card} \mathcal{E}_+(\Gamma^3, \Gamma^3),$$
$$\operatorname{Card} \mathcal{E}(\Gamma^2, \Gamma^3) = 1. \qquad (*)$$

Since Γ^2 is integral, this means that it has at least two distinct southwest corners $(i,j), (k,l)$. Let (r,s) be a southwest corner of Γ^3. Then the pairs $\{(i,j)\}, \{(r,s)\}$ and $\{(k,l)\}, \{(r,s)\}$ are in $\mathcal{E}(\Gamma^2, \Gamma^3)$. Hence, $\operatorname{Card} \mathcal{E}(\Gamma^2, \Gamma^3) \geq 2$ which contradicts $(*)$. □

Remark 6.5. *In* [4], *it is shown that there are no almost principal nilpotent pairs for $\mathfrak{g} = E_6$, E_7, E_8 or F_4 and it is suggested that almost principal nilpotent pairs do not exist in the simply-laced case. It is now clear by Theorem 6.4 that:*

Corollary 6.6. *Almost principal nilpotent pairs do not exist in the simply-laced case.*

This is rather strange and it would be nice to have a more natural explanation.

Theorem 6.7. *Let $\mathfrak{g}$ be of type B_n. Then there is a one-to-one correspondence between the set of conjugacy classes of almost principal nilpotent pairs and the set of pairs of centrally symmetric skew diagrams (Γ^1, Γ^2) satisfying:*

a) $\operatorname{Card} \Gamma^1 + \operatorname{Card} \Gamma^2 = 2n + 1$.
b) Γ^1 *is integral and* Γ^2 *is non integral.*

c) We have either

i) Γ^1 *is near rectangular of type a) or b) and* $\Gamma^2 = \emptyset$*;*

ii) or $\Gamma^1 = \{(0,0)\}$ *and* Γ^2 *is rectangular.*

Proof. Let $\mathbf{e}$ be an almost principal nilpotent pair, and let (Γ^1, Γ^2) be the associated pair of skew diagrams as given in Theorem 1.5. We have 2 cases:

1. Γ^1 or Γ^2 is empty;
2. Γ^1 and Γ^2 are both non-empty.

In the first case, by Proposition 4.5, we have condition c)i). Let us show that in the second case, we have c)ii). So let us assume that Γ^1 and Γ^2 are both non-empty.

By Theorem 3.5 and Proposition 3.11, we must have $\operatorname{Card} \mathcal{E}(\Gamma^1, \Gamma^2) = 1$ and both Γ^1 and Γ^2 must be rectangular for otherwise one of them will have two distinct southwest corners and one can use the argument in the proof of Theorem 6.4 for type D to obtain a contradiction.

Now if C_1 (resp. C_2) is a skew subdiagram of Γ^1 (resp. Γ^2) such that C_1 and C_2 are of the same shape, then one sees easily from the fact that the Γ^i's are centrally symmetric that the pair (C_1, C_2) is in $\mathcal{E}(\Gamma^1, \Gamma^2)$. It follows that if $\operatorname{Card} \mathcal{E}(\Gamma^1, \Gamma^2) = 1$, then there is only one such pair. Now Γ^2 is non integral, therefore Γ^2 contains the square $\{\pm 1/2, \pm 1/2\}$, hence Γ^1 and Γ^2 must satisfy condition c)ii).

Finally, given (Γ^1, Γ^2) satisfying a), b) and c). It is clear that the corresponding distinguished nilpotent pair is almost principal. □

Theorem 6.8. *Let* $\mathfrak{g}$ *be of type* C_n*. Then there is a one-to-one correspondence between the set of conjugacy classes of almost principal nilpotent pairs and the set of pairs of centrally symmetric skew diagrams* (Γ^1, Γ^2) *satisfying:*

a) $\operatorname{Card} \Gamma^1 + \operatorname{Card} \Gamma^2 = 2n$.

b) $\Gamma^1 \subset \mathbb{Z}^2 + (0, 1/2)$ *and* $\Gamma^2 \subset \mathbb{Z}^2 + (1/2, 0)$.

c) We have either

i) Γ^1 *is near rectangular and* $\Gamma^2 = \emptyset$ *or vice versa;*

ii) or Γ^1 *and* Γ^2 *are both (non-empty) rectangular with* $\Gamma^1 \subset \{0\} \times (\mathbb{Z} + 1/2)$ *and* $\Gamma^2 \subset (\mathbb{Z} + 1/2) \times \{0\}$.

Proof. The proof is similar to the one for type B. The only difference is that Γ^1 and Γ^2 are semi-integral which gives condition c)ii). □

7. Centralizer of almost principal nilpotent pairs

In [4], almost principal nilpotent pairs satisfying condition c)i) (resp. condition c)ii)) of Theorem 6.7 or Theorem 6.8 are called of $\mathbb{Z}$-type (resp. non-$\mathbb{Z}$-type). He proved that the centralizer of an almost nilpotent pair of non-$\mathbb{Z}$-type is abelian and suggested the same to be true for an almost nilpotent pair of $\mathbb{Z}$-type. We shall prove in this section that this is indeed true.

Proposition 7.1. *Let* $\mathbf{e}$ *be a distinguished nilpotent pair in a classical simple Lie algebra* $\mathfrak{g}$. *Then there exists an abelian subalgebra in* $Z_+(\mathbf{e})_{int}$ *of dimension* $\operatorname{rk}\mathfrak{g}$.

Proof. Let Γ be a skew diagram (resp. centrally symmetric) and $C \in \mathcal{E}_{p,q}(\Gamma)$ (resp. $(C, \widetilde{C} \in \mathcal{E}_{p,q}(\Gamma,\Gamma))$, $C' \in \mathcal{E}_{p',q'}(\Gamma)$ where $p,q,p',q' \geq 0$ (resp. $(C', \widetilde{C}' \in \mathcal{E}_{p,q}(\Gamma,\Gamma))$.

Let $(i,j) \in C$ be such that $(i,j)+(p,q) \in C'$ (resp. $\in \widetilde{C}'$), then by the definition of a skew subdiagram and the fact that $p,q \geq 0$, we have $(i,j) \in C'$ (resp. $\in \widetilde{C}'$).

Now let $\Gamma, \widetilde{\Gamma}$ be integral centrally symmetric skew diagrams such that $\Gamma \cap \widetilde{\Gamma} = \{(0,0\}$ and $\widetilde{\Gamma} \neq \{(0,0)\}$. By the proof of Theorem 3.9, there exists a pair $(C, \widetilde{C}) \in \mathcal{E}_{p,q}(\Gamma, \widetilde{\Gamma})$, $p,q \geq 0$ such that $C = \{(i,j)\}$ and $\widetilde{C} = \{(k,l)\}$ are southwest corners of Γ and $\widetilde{\Gamma}$ respectively. Thus neither $(i,j)+(p,q)$ or $(k,l)+(p,q)$ is contained in C' or $\widetilde{C}'$ for any pair $(C', \widetilde{C}') \in \mathcal{E}_+(\Gamma,\Gamma) \cup \mathcal{E}_+(\widetilde{\Gamma},\widetilde{\Gamma})$.

Now one verifies easily from Theorem 2.9 that the subalgebra spanned by the elements x_C or $x_{C,C'}$ where

Type A – $C \in \mathcal{E}_+(\Gamma) \setminus \mathcal{E}_{0,0}(\Gamma)$,

Types B or C – $(C,C') \in \mathcal{E}_+(\Gamma^1,\Gamma^1) \cup \mathcal{E}_+(\Gamma^2,\Gamma^2)$,

Type D – $(C,C') \in \mathcal{E}_+(\Gamma^1,\Gamma^1) \cup \mathcal{E}_+(\Gamma^2,\Gamma^2) \cup \mathcal{E}_+(\Gamma^3,\Gamma^3)$ and the pair in $\mathcal{E}_+(\Gamma^2,\Gamma^3)$ as given in the proof of Theorem 3.9 which was described above,

form an abelian subalgebra of $Z(\mathbf{e})$. By Theorem 3.2 and Theorem 3.5, its dimension is $\operatorname{rk}\mathfrak{g}$. □

Theorem 7.2. *Let* $\mathbf{e}$ *be an almost princiapl nilpotent pair in a classical simple Lie algebra* $\mathfrak{g}$. *Then* $Z_+(\mathbf{e})$ *is abelian.*

Proof. By the classification (Theorems 6.4, 6.7 and 6.8) and Proposition 7.1, this is just a straightforward case by case computation. □

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